Matematyka Dyskretna

Ćwiczenia – Lista 2

Zadanie 3.

(a) liczba warunków początkowych: 2 (nie ma możliwości obliczenia a_0 i a_1),

(b) liczba warunków początkowych: 3 (nie ma możliwości obliczenia a_0, a_1 i a_2),

(c) wzór jest tak skonstruowany, że nie potrzeba określać jawnie żadnego warunku początkowego,

Zadanie 4.

(a)

$$f_n = f_{n-1}+3^n = 3^n+3^{n-1}+f_{n-2} = 3^n+3^{n-1}+\cdots+3^{n-k}+f_{n-k-1} = 3^n+3^{n-1}+\cdots+3^2+3 = \sum_{i=1}^{n} 3^i$$

(b)

$$\begin{aligned} h_n &= h_{n-1} + (-1)^{n+1}n = (-1)^{n+1}n + (-1)^n(n-1) + h_{n-2} = \\ &= (-1)^{n+1}n + (-1)^n(n-1) + (-1)^{n-1}(n-2) + \cdots + (-1)^{n-k}(n-k-1) + h_{n-k-2} = \\ &= (-1)^{n+1}n + (-1)^n(n-1) + \cdots + (-1)^2 \cdot 1 = \sum_{i=1}^{n}(-1)^{i+1} \cdot i \end{aligned}$$

(c)

$$\begin{aligned} l_n &= l_{n-1} \cdot l_{n-2} = (l_{n-2})^2 \cdot l_{n-3} = (l_{n-3})^3 \cdot (l_{n-4})^2 = \\ &= (l_{n-4})^5 \cdot (l_{n-6})^3 = \cdots = (l_2)^{F_{n-1}} \cdot (l_1)^{F_{n-2}} = 2^{F_{n-1}+F_{n-2}} = 2^{F_n} \end{aligned}$$

gdzie F_n jest ciągiem Fibonacciego o wzorze jawnym $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

Zadanie 5.

(a) $a_0 = 1$, $a_n = 2/a_{n-1}$,

$$a_0 = 1,\ a_1 = 2,\ a_2 = 1,\ a_3 = 2,\ a_4 = 1,\ \ldots$$

Zatem:

$$a_n = (n+1) \bmod 2 + (-1)^{n+1} + 1$$

Dowód.

- dla n parzystych:

$$(n+1) \bmod 2 = 1,\ (-1)^{n+1} = -1,\ \text{więc } a_n = 1$$

- dla n nieparzystych:

$$(n+1) \bmod 2 = 0,\ (-1)^{n+1} = 1,\ \text{więc } a_n = 2$$

□

(b) $b_0 = 0,\ b_n = 1/(1 + b_{n-1})$,

$$b_0 = 0,\ b_1 = 1,\ b_2 = \frac{1}{2},\ b_3 = \frac{2}{3},\ b_4 = \frac{3}{5},\ b_5 = \frac{5}{8},\ \ldots$$

Zatem:

$$b_n = \frac{F_n}{F_{n+1}},\quad \text{gdzie } F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\quad (F_n \text{ – ciąg Fibonacciego})$$

Dowód.

$$b_n = \frac{1}{1 + \frac{F_{n-1}}{F_n}} = \frac{1}{\frac{F_n + F_{n-1}}{F_n}} = \frac{F_n}{F_n + F_{n-1}} = \frac{F_n}{F_{n+1}}$$

□

(c) $c_0 = 1,\ c_n = \sum_{i=0}^{n-1} c_i$,

$$c_n = \sum_{i=0}^{n-1} c_i = c_{n-1} + \sum_{i=0}^{n-2} c_i = \sum_{i=0}^{n-2} c_i + \sum_{i=0}^{n-2} c_i = c_{n-2} + \sum_{i=0}^{n-3} c_i + c_{n-2} + \sum_{i=0}^{n-3} c_i =$$
$$= 4 \cdot \sum_{i=0}^{n-3} c_i = 8 \cdot \sum_{i=0}^{n-4} c_i = 2^4 \cdot \sum_{i=0}^{n-5} c_i = 2^5 \cdot \sum_{i=0}^{n-6} c_i = 2^{k-1} \cdot \sum_{i=0}^{n-k} c_i =$$
$$= 2^{n-2} \cdot \sum_{i=0}^{1} c_i = 2^{n-2} \cdot (c_0 + c_1) = 2^{n-2} \cdot 2 = 2^{n-1} \text{ dla } n > 0$$

Zatem:

$$c_n = \begin{cases} 1 & \text{dla } n = 0 \\ 2^{n-1} & \text{dla } n \in \mathbb{N} \end{cases}$$

(d) $d_0 = 1,\ d_1 = 2,\ d_n = (d_{n-1})^2/d_{n-2}$,

$$d_0 = 1,\ d_1 = 2,\ d_2 = 4,\ d_3 = 8,\ d_4 = 16,\ \ldots$$

Zatem:

$$d_n = 2^n$$

Dowód.

$$d_n = \frac{(2^{n-1})^2}{2^{n-2}} = \frac{2^{2n-2}}{2^{n-2}} = \frac{2^{2n}}{4} \cdot \frac{4}{2^n} = \frac{2^{2n}}{2^n} = 2^n$$

□

Zadanie 9.

(a) $f(1) = 1,\ f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + 1$,

Wzór jawny:

$$f(n) = 2n - 1$$

Dowód.

1. $f(1) = 2 \cdot 1 - 1 = 1$ ✓
2. Załóżmy, że $n \geq 2$ oraz dla wszystkich $k < n$ zachodzi $f(k) = 2k - 1$. Zatem:

$$f(n) = 2\left\lceil \frac{n}{2} \right\rceil - 1 + 2\left\lfloor \frac{n}{2} \right\rfloor - 1 + 1 = 2\left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor\right) - 1 = 2n - 1$$

□

(b) $g(0)=0,\ g(n)=g\left(\left\lfloor \frac{n}{2}\right\rfloor\right)+\lfloor \log_2 n\rfloor,$

Początkowe wyrazy ciągu:

$$\begin{array}{rcccccccccccccccccc} g(n): & 0, & 1, & 1, & 3, & 3, & 3, & 3, & 6, & 6, & 6, & 6, & 6, & 6, & 6, & 6, & \overbrace{10}^{16\times}, & \overbrace{15}^{32\times}, & \overbrace{21}^{64\times}, \dots \\ n: & 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & 11, & 12, & 13, & 14, & 15, & 31, & 63, \dots \end{array}$$

Zatem:

$$g(n)=\sum_{i=0}^{\lfloor \log_2 (n+1)\rfloor} i$$

Dowód.

1. $g(0)=\sum_{i=0}^{\lfloor \log_2 1\rfloor} i=\sum_{i=0}^{0} i=0 \qquad \surd$
2. Załóżmy, że $n\geq 1$, oraz dla wszystkich $k<n$ zachodzi:

$$g(k)=\sum_{i=0}^{\lfloor \log_2 (k+1)\rfloor} i$$

więc:

$$g(n)=\sum_{i=0}^{\left\lfloor \log_2\left(\left\lfloor \frac{n}{2}\right\rfloor+1\right)\right\rfloor} i+\lfloor \log_2 n\rfloor=\sum_{i=0}^{\lfloor \log_2 (n+1)\rfloor} i$$

Uzasadnienie:

- $\log_2 n=i \ \Leftrightarrow\ 2^i=n$, oraz $2^{i-1}=\frac{2^i}{2}$, zatem $2^{i-1}=\frac{n}{2}\ \Leftrightarrow\ \log_2\frac{n}{2}=i-1$
- $\sum_{i=0}^{n} i=\sum_{i=0}^{n-1} i+n$

□

Zadanie 13.

Liczby harmoniczne: $H_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$ $(\star)$

spełniają zależność rekurencyjną $H_n=\frac{1}{n}\left(H_{n-1}+H_{n-2}+\cdots+H_1\right)+1$ $(\star\star)$ dla $n>1$.

Dowód.

1.

$$H_1=1 \qquad H_2=1+\frac{1}{2}=\frac{1}{2}\cdot H_1+1=\frac{1}{2}\cdot 1+1=1\frac{1}{2} \qquad \surd$$

2.

$$H_{n+1} = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) + \frac{1}{n+1} \qquad (\star)$$

$$H_{n+1} = H_n + \frac{1}{n+1}$$

$$H_{n+1} = \frac{1}{n}\left(H_{n-1} + H_{n-2} + \dots + H_1\right) + 1 + \frac{1}{n+1} \quad / \cdot (n+1) \qquad (\star\star)$$

$$(n+1)H_{n+1} = \frac{n+1}{n}\left(H_{n-1} + H_{n-2} + \dots + H_1\right) + n + 1 + 1$$

$$(n+1)H_{n+1} = \left(H_{n-1} + H_{n-2} + \dots + H_1\right) + \underbrace{\frac{1}{n}\left(H_{n-1} + H_{n-2} + \dots + H_1\right) + 1}_{H_n} + n + 1$$

$$(n+1)H_{n+1} = \left(H_n + H_{n-1} + H_{n-2} + \dots + H_1\right) + (n+1) \quad / : (n+1)$$

$$H_{n+1} = \frac{1}{n+1}\left(H_n + H_{n-1} + H_{n-2} + \dots + H_1\right) + 1$$

□

Zadanie 14.

(a) $F_0 + F_1 + F_2 + \dots + F_n = F_{n+2} - 1$,

Dowód.

$$\underbrace{F_0 + F_1 + F_2 + \dots + F_n}_{F_{n+2}-1} + F_{n+1} = F_{n+3} - 1$$

$$F_{n+2} - 1 + F_{n+1} = F_{n+3} - 1$$

$$F_{n+1} + F_{n+2} = F_{n+3}$$

□

(b) $F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n}$, dla $n \geq 1$,

Dowód.

$$\underbrace{F_1 + F_3 + F_5 + \dots + F_{2n-1}}_{F_{2n}} + F_{2n+1} = F_{2n+2}$$

$$F_{2n} + F_{2n+1} = F2n + 2$$

□

(c) $F_0^2 + F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$,

Dowód.

$$\underbrace{F_0^2 + F_1^2 + F_2^2 + \dots + F_n^2}_{F_n F_{n+1}} + F_{n+1}^2 = F_{n+1} F_{n+2}^2$$

$$F_n F_{n+1} + F_{n+1}^2 = F_{n+1} F_{n+2}^2$$

$$F_{n+1}(F_n + Fn + 1) = F_{n+1} F_{n+2}^2$$

$$F_{n+1} F_{n+2}^2 = F_{n+1} F_{n+2}^2$$

□

(d) $F_n F_{n+2} = F_{n+1}^2 + (-1)^{n+1}$

Dowód.
Aby udowodnić prawdziwość powyższego równania pokażę, że zachodzi następująca równość:

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1} \qquad (\star)$$

Lemat:

$$(-1)^{k+1} = (-1)^{k+3}, \text{ ponieważ } (-1)^{k+3} = (-1)^{k+1} \cdot (-1)^2 = (-1)^{k+1} \cdot 1 = (-1)^{k+1}$$

Załóżmy, że $n \geq 0$ oraz dla wszystkich $k < n$ zachodzi:

$$F_k F_{k+2} - F_{k+1}^2 = (-1)^{k+1}$$

Pokażmy, że powyższa równość jest również prawdziwa dla $k+2$:

$$\begin{aligned} F_{k+2}F_{k+4} - F_{k+3}^2 &= F_{k+2}(F_{k+2} + F_{k+3}) - (F_{k+1} + F_{k+2})^2 = \\ &= F_{k+2}(F_{k+2} + F_{k+2} + F_{k+1}) - F_{k+1}^2 - 2F_{k+1}F_{k+2} - F_{k+2}^2 = \\ &= F_{k+2}(F_k + 2F_{k+1} + Fk + 2) - F_{k+1}^2 - 2F_{k+1}F_{k+2} - F_{k+2}^2 = \\ &= \underline{F_{k+2}^2} + \underline{2F_{k+1}F_{k+2}} + F_{k+2}F_k - F_{k+1}^2 - \underline{2F_{k+1}F_{k+2}} - \underline{F_{k+2}^2} = \\ &= F_kF_{k+2} - F_{k+1}^2 \overset{\text{z założenia indukcyjnego}}{=} (-1)^{k+1} \overset{\text{na mocy lematu}}{=} (-1)^{k+3} \end{aligned}$$

□

Zadanie 15.

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Dowód.

1. $F_0 = \frac{1}{\sqrt{5}}(1-1) = 0 \quad \surd$

   $F_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2} \right) = 1 \quad \surd$

2. Załóżmy, że wzór jest prawdziwy dla wszystkich $k \leq n+1$ ($k \in \mathbb{N}$). Pokażę, że jest też prawdziwy dla $k = n+2$.

$$\begin{aligned} F_{n+2} = F_{n+1} + F_n &= \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} + \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) = \\ &= \frac{1}{\sqrt{5}} \left(\left(1 + \frac{1+\sqrt{5}}{2} \right) \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(1 + \frac{1-\sqrt{5}}{2} \right) \left(\frac{1-\sqrt{5}}{2} \right)^n \right) = \\ &= \frac{1}{\sqrt{5}} \left(\left(\frac{3+\sqrt{5}}{2} \right) \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{3-\sqrt{5}}{2} \right) \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \overset{\star}{=} \\ &\overset{\star}{=} \underline{\frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+2} \right)} \end{aligned}$$

$\star \quad \frac{3\pm\sqrt{5}}{2} = \frac{6\pm2\sqrt{5}}{4} = \frac{5\pm2\sqrt{5}+1}{4} = \left(\frac{\sqrt{5}\pm1}{2} \right)^2$

□